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Inventory System with Postponed Demands and Service Facilities-MAM Approach

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Abstract

In this paper, we consider an (s,S) *inventory system with random service time Primary demands occur according to Poisson process with parameter* λ*. In this system, there is a finite buffer whose capacity varies according to the inventory level at any given time. When the maximum buffer size is reached, further demands join a pool of infinite capacity with probability* Υ *and with probability* $(1-\Upsilon)$ *it is lost forever. When inventory level is larger than the number of customers in the buffer, an external demand can enter the buffer for service. A pooled customer is transferred to the buffer for service at a service completion epoch with probability* p *if the inventory exceeds* $s+1$ *and provided the number of customer in the buffer is less than the nimber of items held in the inventory. It is assumed that initially the inventory level is* S*. When inventory level reaches* s *an order for replenishment is placed. The lead time is exponentially distributed with parameter* β*. We obtain the steady state system size distribution. Some performance measures are obtained and a few numerical illustrations provided.*

Key words*: Postponed demand, Finite buffer, Replenishment, Service facilities.*

1. Introduction

In most of the inventory models it is assumed that the inventory deplete at a rate equal to demand rate. However, it becomes unrealistic for the service facilities where the stocked item is delivered to the customers after some service is performed. In this paper we consider an (s,S) inventory system with service facilities. Arrival of demands form a Poisson process with parameter $\lambda(>0)$ to a buffer of finite capacity equal o the inventory level at any given time t. When the maximum buffer size is reached, further demands join a pool of infinite capacity, with probability γ or with probability $(1-\gamma)$ it is lost forever, Pooled customers are taken for service at a service completion epoch if the inventory level is at least $s+1$. The service time is assumed to be exponentially distributed with parameter μ. It is also assumed that initially the inventory level is S. When inventory level reaches s due to service, an order for replenishment is placed. The lead time is exponentially distributed with parameter β.

The literature available on inventory system with service facilities is too limited. Berman, Kim and Shimshak [2] consider an inventory system with service in which they assume that both the demand and the service rate are deterministic and constant and as such queues can form only during stock out period. They determine optimal order quantity the minimize the total cost per unit time. Later Berman an Kim [3, 4] analyze the non-deterministic inventory model for service facilities. They analyze the system in which customers arrive at a service facility according to Paisson process where service times are exponentially distributed and each customer demands exactly one unit of the item in the inventory both zero and positive lead time cases are discussed. Krishnamoorthy et al [9] analyze a retrial production inventory system with service time in which primary demands occur according to Markovian Arrival Process (MAP). Using matrix analytic method they carry out the steady state analysis of the system and some performance measures and obtained. Berman and Sapna [5, 6] investigate inventory control at a service facility, which uses one item if inventory for service provided. Assuming Poisson arrival process, arbitrarily distributed service times and zero lead times they analyze the system with the restriction that, waiting space is finite. Under a specific cost structure they derive the optimum ordering quantity that minimizes the long run expected cost rate. Arivarignan, Elango and Arumugam [1] consider a perishable inventory system with service facility where arrival times of customers form a Poisson process. Each customer requires a single item which is delivered through a service of random duration having exponential distribution. Krishnamoorthy and Mohammad Islam [7, 8] analyze an inventory system with postponed demands both, for perishable and non-perishable inventoried items. With demands forming a Poisson process they analyze the model under the additional assumptions, when inventory level reaches zero due to demands, further demands are sent to a pool of capacity $M\ (< \infty)$. Demand to the pooled customers are satisfied only after replenishment against the order placed on reaching the level s. Further they are served only if the inventory level is at least $s + 1$. They study the system both in transient and steady state cases. Mohammad Ekramol Islam(2012) further improved the model in this direction and considered (s,S) Inventory system with postponed demands at a service facility

1.1. The Notations

The notations used in the sequel are explained below:

$I(l)$=Inventory level at time l; this takes value $\{0, 1, \dots, S\}$

$B(t)$=Number of customers in the buffer at time t

$N(t)$=Number of customers in the pool at time t

A'-Transpose of a matrix A

e-Denote the column vector of I's

2. The Steady State Analysis of the Model

$\{(N(t), I(l), B(l)), t \geq 0\}$ is a continuous time Markov chain with state space given by

$$\Omega = \{i, j, k);\ i \geq 0, 0 \leq j \leq S, 0 \leq k \leq j\}$$

These states are arranged in the lexicographic order. Define the following auxiliary matrices for use in sequel.

(1) $$B_0 = \begin{bmatrix} A_{00} & & & \tilde{A}_{10,Q} & & & \\ \tilde{A}_{10} & \tilde{A}_{11} & & & \tilde{A}_{10,Q+1} & & \\ & & \ddots & & \ddots & & \\ & & & & & \tilde{A}_{S-1,S-1} & \\ & & & & & \tilde{A}_{S,S-1} & \tilde{A}_{S,S} \end{bmatrix}$$

Where

(2) $$A_{00} = -\lambda\gamma - \beta$$

(3) $$\tilde{A}_{11} = \begin{bmatrix} -\lambda - \beta & \lambda \\ 0 & -\lambda\gamma - \beta - \mu \end{bmatrix}$$

(4) $$\tilde{A}_{ii} = \begin{bmatrix} -\lambda - \beta\beta & & \\ & -\lambda - \beta - \mu & \lambda \\ & \ddots & \ddots \\ & & -\lambda\gamma - \beta - \mu \end{bmatrix} for\ 2 \leq i \leq s$$

(5) $$\tilde{A}_{ii} = \begin{bmatrix} -\lambda & \lambda & & \\ & -\lambda - \mu & \lambda & \\ & \ddots & \ddots & \ddots \\ & & -\lambda\gamma - \mu & \end{bmatrix} for + 1 \leq i \leq s$$

(6) $$\tilde{A}_{1,0} = \begin{bmatrix} 0 \\ \mu \end{bmatrix}$$

(7) $$\tilde{A}_{i,i-1} = \begin{bmatrix} 0 \\ I_i \end{bmatrix} for\ 2 \leq i \leq s$$

(8) $$\tilde{A}_{0,Q} = [\beta, 0, \dots, 0]$$

(9) $$\tilde{A}_{i,i+Q} = [I_{i+1}, 0, 0, \dots, 0]\beta\ for\ 1 \leq i \leq s$$

$$A_1 = \begin{bmatrix} A_{00} & & & \tilde{A}_{0,Q} & & & \\ \tilde{A}_{10} & \tilde{A}_{11} & & & \tilde{A}_{1,Q+1} & & \\ & & \ddots & & \ddots & & \\ & & & & & \tilde{A}_{S-1,S-1} & \\ & & & & & \tilde{A}_{S,S-1} & \tilde{A}_{S,S} \end{bmatrix} \tag{10}$$

Where

$$\tilde{A}_{i,i-1} = \begin{bmatrix} 0 \\ I_i \end{bmatrix} for\ 2 \le i \le s \tag{11}$$

$$\tilde{A}_{i,i-1} = \begin{bmatrix} 0 \\ I_i \end{bmatrix} \big(\mu(1-p)\big)\ for\ s+1 \le i \le S \tag{12}$$

$$\tilde{A}_2 = \begin{bmatrix} C_{00} & & & & \\ \tilde{C}_{10} & \tilde{C}_{11} & & & \\ & \tilde{C}_{21} & \tilde{C}_{22} & & \\ & & \ddots & \ddots & \\ & & & \tilde{C}_{ss-1} & \tilde{C}_{ss} \end{bmatrix} \tag{13}$$

Where

(14) $C_{00} = 0$

(15) $\tilde{C}_{ii}$ are matrices of all elements with zeros for $s+1 \le i \le S$

(16) $\tilde{C}_{ii-1}$ are matrices of all elements with zeros for $1 \le i \le S$

$$\tilde{A}_{ii-1} = \begin{bmatrix} 0 & & & & \\ & p\mu & & & \\ & & p\mu & & \\ & & & \ddots & \\ & & & & p\mu \\ & & & & 0 \end{bmatrix} for\ s+1 \le i \le S \tag{17}$$

where $\tilde{C}_{ii-1}$ is a $(i\ +\ 1) \times 1$ matrix

$$A_0 = \begin{bmatrix} B_{00} & & & \\ & \tilde{B}_{11} & & \\ & & \ddots & \\ & & & \tilde{B}_{SS} \end{bmatrix} \tag{18}$$

(19) $B00\ =\ \lambda\gamma$

(20) $\tilde{B}_{ii} = (a_{i+1},\ a'_{i+1})\lambda\gamma$ 1 ≤ i ≤ S

Where $ai + 1$ is a column vector of zeros except last entry which is 1

$\tilde{B}_{ii}$ is the $(i\ +\ 1) \times (i\ +\ 1)$ matrix.

The Markov chain $\{(N(t), I(t), B(t)), t \ge 0\}$ has the generator Q in partitioned from given by

$$Q = \begin{bmatrix} B_0 & A_0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

where, the sentries in Q are given by (1) to (20).

Let $A\ =\ A_0 + A_1 + A_2$ and π denote the steady-state probability vector of A, i,e,

$$\pi A\ =\ 0,\ \pi e\ =\ 1$$

The vector π can be partitioned as

$$\pi(i)\ =\ (\pi(i,0), (i,1), \dots\dots, \pi(i,i));\ i\ =\ 0, 1, \dots\dots.S$$

Then the π's can be calculate as

$$\pi(s - i)\ =\ \pi(s)\beta_{s-i}$$

$$\pi(S - i) = \pi(s)\beta_{s-i}$$

Where

$$\beta_{(s-i)} = \begin{bmatrix} -\beta_{s-i+1}\left(\tilde{A}_{s-i+1,s-i} + \tilde{C}_{s-i+1,s-i}\right)\left(\tilde{A}_{s-i,s-i} + \tilde{B}_{s-i,s-i} + \tilde{C}_{s-i,s-i}\right) & if\ i = 1, 2, \dots, s \\ I & if\ i = 0 \end{bmatrix}$$

$$\beta_{(S-i)} = \begin{bmatrix} \left[-\beta_{S-1+1}\tilde{A}_{S-i+1,S-i} - \beta_{S-i}\tilde{A}_{S-i,S-i}\right]\left(\tilde{A}_{S-i,S-i} + \tilde{B}_{S-i,S-i} + \tilde{C}_{S-i,S-i}\right)^{-1} & if\ i = 1, 2, \dots, s \\ -\tilde{A}_{sS}\left(\tilde{A}_{S,S} + \tilde{B}_{S,S} + \tilde{C}_{S,S}\right)^{-1} & if\ i = 0 \end{bmatrix}$$

$$\pi(i) = \beta_{i+1}\left(A_{i+1,i} + C_{i+1,i}\right)\left(A_{i,i} + B_{i,i} + C_{i,i}\right)^{-1}\ if\ I = Q - 1, Q - 2, \dots, s + 1$$

We have the following result on system stability

Lemma 2.1.

$$\sum_{i=s+2}^{S}\sum_{j=1}^{i-1}\pi(i,j) > \left(\frac{\lambda\gamma}{p\mu}\right)\sum_{i=0}^{s}\pi(i,i)$$

Proof. From the well-known result [Neuts (1981) theorem 1.7.1] on the positive recurrence of Q, which states that-

$$\pi A_0 e < \pi A_2 e$$

and by exploiting the structure of the matrices A_0 and A_2, the stated result follows.

Theorem 2.2. When the stability condition holds the steady state probability vector x of Q which satisfies $xQ = 0, x\,e = 1$ exists.

The steady state probability vector

$$x = (x(0), x\,(1), x(2), \dots)$$

where components are given by

$$x(i) = x(0)\,Ri, i \geq 0$$

Where R is the minimal non-negative solution of the matrix quadratic equation:-

$$R^2A_2 + RA_1 + A_0 = 0$$

The vector x(0) can be calculated using the equation

$$x(0)[B_0 + RA_2] = 0$$

together with the normalizing condition

$$x(0)(1 - R)^{-1}e = 1$$

Proof. Follows immediately from the well-known result on matrix-geometric methods (see Neuts (1981).

For calculating the rate matrix R one can use Logarithmic Reduction Algorithm

3. System Performance Measures

Steady state probability vector $x = (x(0), x\,(1), x(2), \dots)$ can be partitioned as

$$x\,(i) \;=\; (y(i,j,k));\; i \geq 0,\, 0 \leq j \leq S,\, 0 \leq k \leq j$$

Some of the system performance measures are given below:

(1) The probability mass function of number of customer in the pool: The probability that there are I customers in the pool is given by

$$P_i = x(i)e = x(0)R^i e;\, i\geq 0$$

(2) Expected Inventory level in the system: Expected inventory level in the system is given by-

$$\alpha_1 = \sum_{i=0}^{\infty}\left[\left\{\sum_{j=1}^{S} j \sum_{k=0}^{j} y(i,j,k)\right\}\right] e$$

(3) Expected number of customers in the buffer is,

$$\alpha_2 = \sum_{i=0}^{\infty}\left[\sum_{j=1}^{S} j \sum_{k=1}^{j} k y(i,j,k)\right] e$$

(4) Expected number of customers in the pool,

$$\alpha_3 = \sum_{i=0}^{\infty} i x(i) e = x(0)R(I-R)^{-2}\, e$$

(5) Average customers lost to the system is,

$$\alpha_4 = \lambda(1-\gamma)\sum_{i=0}^{\infty}\left[\sum_{j=k=0}^{S} y(i,j,k)\right] e$$

(6) Expected rate that a customer will enter the pool is,

$$\alpha_5 = \lambda\gamma\sum_{i=0}^{\infty}\left[\sum_{j=k=0}^{S} y(i,j,k)\right] e$$

(7) The average rate at which the pooling customers will enter the buffer is given by

$$\alpha_6 = p\mu\sum_{i=0}^{\infty}\left[\sum_{j=s+1}^{S}\sum_{k=1}^{j} y(i,j,k)\right] e$$

4. Cost Function

Define

$C_1 =$ Inventory holding cost of the system, $C_2 =$ Cost for buffer customers in the system

$C_3 =$ Cost for pool customers in the system, $C_4 =$ Cost of customers lost to the system, So the total expected cost of the system is

$$E(TC(S,s,P)) = C_1\alpha_1 + C_2\alpha_2 + C_3\alpha_3 + C_4\alpha_4$$

$$E(TC(S,s,P)) = C_1 \textstyle\sum_{i=0}^{\infty}\left[\left\{\sum_{j=1}^{S} j \sum_{k=0}^{j} y(i,j,k)\right\}\right] e + C_2 \sum_{i=0}^{\infty}\left[\sum_{j=1}^{S} j \sum_{k=1}^{j} k y(i,j,k)\right] e$$

$$+ C_3 \textstyle\sum_{i=0}^{\infty} i x(i) e + \quad + C_3 \sum_{i=0}^{\infty} i x(i) e$$

By using the above cost function, we can exploit a lot of interesting feasure and can make a sensitivity analysis.

4. Numerical Illustration

We provide a simple numerical illustration based on our performance measure by fixing Fixed $S = 5, s = 2, Q = 3, \lambda = 0.5, \mu = 0.7, \beta = 0.6, p = 0.6, \gamma = 0.6, C_1 = 1, C_2 = 2, C_3 = 1,. C_4 = 2.$ In table 1-5, we provide measures of the system performance by fixing the parameter's values involved in the system, we vary over the parameters λ, μ, β, p and γ. For different values of these parameters corresponding values of the system measures are provided.

Table: 1. Arrival rate Vs Different performance measures

	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$	$\lambda = 0.6$
α_1	3.54122	3.44974	3.27038	3.1189	3.00847
α_2	0.362165	0.593453	0.85437	1.14709	1.47747
α_3	0.0427584	0.176074	0.551752	1.71793	9.46571
α_4	0.00222627	0.00937859	0.0246719	0.050412	0.0882317
α_5	0.3394	0.0140679	0.0370079	0.075618	0.132348
α_6	0.103909	0.143281	0.175928	0.203971	0.229517

Table: 2. Service rate Vs Different performance measures

	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$	$\mu = 0.9$	$\mu = 1.0$
α_1	3.15381	3.1189	3.10037	3.09042	3.0852
α_2	1.44111	1.14709	0.945552	0.800566	0.692144
α_3	3.87896	1.71793	1.07129	0.778464	0.617492
α_4	0.0634823	0.050412	0.0419241	0.0361165	0.0319699
α_5	0.0952251	0.075618	0.0628862	0.0541748	0.0479548
α_6	0.125296	0.146179	0.167062	0.207452	0.208827

Table: 3. Replenishment rate Vs Different performance measures

	$\beta = 0.4$	$\beta = 0.5$	$\beta = 0.6$	$\beta = 0.7$	$\beta = 0.8$
α_1	2.7194	2.9549	3.1189	3.24018	3.33281
α_2	1.10445	1.12528	1.14709	1.16721	1.18511
α_3	6.50534	2.68521	1.17793	1.29464	1.06335
α_4	0.0695216	0.0576941	0.050412	0.0456062	0.042261
α_5	0.104282	0.0865412	0.075618	0.0684092	0.0633917
α_6	0.178756	0.193112	0.203971	0.212473	0.219313

Table: 4. Probability of transferring the pool customer to buffer Vs Different performance measures

	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$
α_1	3.1357800	3.12578	3.11890	3.11374	3.06936
α_2	1.1719900	1.15660	1.14709	1.14089	1.00008
α_3	4.2286200	2.41191	1.17793	1.35453	0.68010
α_4	0.0505280	0.05041	0.05041	0.05048	0.04668
α_5	0.0757921	0.07562	0.07562	0.07571	0.04669
α_6	0.1381150	0.171072	0.20397	0.23680	0.25003

Table: 5. Joining probability to pool Vs Different performance measures

	$\gamma = 0.4$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$
α_1	3.14825	3.13416	3.1189	3.10209	3.08329
α_2	1.05942	1.10032	1.14709	1.20083	1.26299
α_3	0.65343	1.05776	1.17793	2.90313	5.41881
α_4	0.06727	0.059313	0.050412	0.04035	0.02885
α_5	0.04484	0.059312	0.075618	0.094147	0.11539
α_6	0.196428	0.199945	0.203971	0.208600	0.213959

5.Sensitivity Analysis

Table: 6 . Arrival rate Vs Different performance measures

Arrival Rate	Total Cost	Service rate	Total cost	Replenishment	Total cost
0.2	4.3127609	0.6	9.92768646	0.4	11.5726832
0.3	4.83147718	0.7	7.30193400	0.5	8.0052482
0.4	5.5802158	0.8	6.1466122	0.6	6.9618340
0.5	7.231834	0.9	5.5422490	0.7	6.9604524
0.6	9.856552	1.0	5.1509190	0.8	6.8586410

Form the table 6, It is observed that the arrival rate is has a vital impact to the system. The result is obvious as the rate is increased it has impact on higher reordering, lost sales and also increased the cost of carrying pool and Buffer customers. The service rate increase indicates cost decrease as because of less cost of customers lost and less waiting time to pool and buffer customers. Finally we can see that replenishment rate increase results cost decrease.This is obvious as more replenish will support more service, less the cost of lost sales as well as less cost for pool and buffer.

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